

Involution on pseudoisotopy spaces and the space of the nonnegatively curved metrics

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Abstract

We prove that certain involutions defined by Vogell and Burghelea-Fiedorowicz on the rational algebraic K-theory of spaces coincide. This gives a way to compute the positive and negative eigenspaces of the involution on rational homotopy groups of pseudoisotopy spaces from the involution on rational S^1 -homology group of the free loop space of a simply-connected manifold. As an application, we give explicit dimensions of the open manifolds V that appear in Belegarde-Farrell-Kapovitch's work for which the spaces of complete nonnegatively curved metrics on V have nontrivial rational homotopy groups.

1 Introduction

In this paper, all spaces of diffeomorphisms, pseudoisotopy spaces and spaces of Riemannian metrics under consideration are equipped with the smooth compact-open topology.

Given a compact smooth manifold M possibly with boundary there is a subgroup $P(M)$ of $\text{Diff}(M \times [0, 1])$ called the *pseudoisotopy space* of M (see Section 3 for the definitions). In order to get the homotopy information of the topological group $\text{Diff}(M, \partial)$ consisting of all self-diffeomorphisms of M which restrict to the identity of the boundary ∂M , one crucial step is to study a canonical involution $\iota : P(M) \rightarrow P(M)$ on this pseudoisotopy space (c.f. [Hat78, Proposition 2.1]).

There is also the *stable pseudoisotopy space* $\mathcal{P}(M)$ satisfying $\pi_k \mathcal{P}(M) \cong \pi_k P(M)$ when $\dim M \gg k$, and the involution on $\pi_* P(M)$ induces an involution on $\pi_* \mathcal{P}(M)$ (see Section 3 for details). This stable pseudoisotopy space turns out to be closely related to the Waldhausen's K-theory of M (c.f. [Wal78]), which can further be analyzed, at least in the simply-connected case, via equivariant S^1 -homology of the free loop space LM of M (c.f. [Wal82, Wal78, Bur86, BF86, Goo85, Goo86, Vog85, BF85, Lod90, Lod96, KS88]).

In this paper we establish formulas for computing the positive and negative eigenspaces of the involution on $\pi_* \mathcal{P}(M) \otimes \mathbb{Q}$ in terms of the involutions on the Waldhausen's K-theory $A(M)$ and the rational S^1 -homology $H_*^{S^1}(LM; \mathbb{Q})$ when the smooth manifold M is simply-connected. To be explicit, for a \mathbb{Q} -vector space V with an involution T , let

$$\text{Inv}_T^\varepsilon V = \{v \in V | T(v) = \varepsilon v\}$$

where $\varepsilon = \pm$ and we use the notation $\text{Inv}^\varepsilon V$ instead of $\text{Inv}_T^\varepsilon V$ when the involution T is clear from the context. Denote $\pi_i(-) \otimes \mathbb{Q}$ by $\pi_i^\mathbb{Q}(-)$. Let $\tau_V : A(M) \rightarrow A(M)$ be the involution defined by Vogell in [Vog85], which induces an involution $\tau_{V*} : \pi_i^\mathbb{Q} A(M) \rightarrow \pi_i^\mathbb{Q} A(M)$. Equip $H_*^{S^1}(LM, *, \mathbb{Q})$ with the geometric involution obtained by “reversing loops” (see Section 2 for the definition.) We establish the following theorem.

Theorem 1.1. *Let M be a simply-connected compact smooth manifold. Then for $i \geq 0$*

$$\begin{aligned} \dim \text{Inv}^+ \pi_i^\mathbb{Q} \mathcal{P}(M) &= \dim \text{Inv}_{\tau_{V*}}^- \pi_{i+2}^\mathbb{Q} A(M) \\ &= \delta_i + \dim \text{Inv}^+ H_{i+1}^{S^1}(LM, *, \mathbb{Q}) \\ \dim \text{Inv}^- \pi_i^\mathbb{Q} \mathcal{P}(M) &= \dim \text{Inv}_{\tau_{V*}}^+ \pi_{i+2}^\mathbb{Q} A(M) - \dim H_{i+2}(M; \mathbb{Q}) \\ &= \dim \text{Inv}^- H_{i+1}^{S^1}(LM, *, \mathbb{Q}) - \dim H_{i+2}(M; \mathbb{Q}) \end{aligned}$$

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where $\delta_i = \begin{cases} 1, & \text{if } i \equiv 3 \pmod{4}; \\ 0, & \text{otherwise.} \end{cases}$

Remark 1.2. By [Dwy80], $\dim \pi_i^{\mathbb{Q}} A(M)$ is finite for all i if M is simply-connected and $\pi_i(M)$ is finitely generated for each i .

Remark 1.3. It had already been suggested by Burghlea in his survey paper [Bur89, Theorem 3.5] that the involution on $\pi_*^{\mathbb{Q}} A(M)$ could be computed from the involution on $H_{S^1}^*(LM; \mathbb{Q})$.

As an application of Theorem 1.1, we give explicit dimensions of the open manifolds that appear in Belegradek-Farrell-Kapovitch's work [BFK15, Theorem 1.2] on which the spaces of complete nonnegatively curved Riemannian metrics have nontrivial rational homotopy groups. For example, we can prove that

$$\pi_{18}^{\mathbb{Q}} \mathcal{R}_{K \geq 0}(TS^4 \times S^{56}) \neq 0$$

where $\mathcal{R}_{K \geq 0}(V)$ is the space of complete nonnegatively curved Riemannian metrics.

The remaining parts of the paper are organized as follows. Sections 2 and 3 aim at proving Theorem 1.1. In Section 2, we present the relation between the involution τ_V on $A(M)$ and the geometric involution on $H_*^{S^1}(LM; \mathbb{Q})$. In Section 3, after reviewing Waldhausen and Vogell's work on the K-theory of spaces, we show the relation between the involution τ_V on $A(M)$ and the involution on $\pi_* \mathcal{P}(M)$, and then we prove Theorem 1.1. In Section 4, we first calculate the involution on the rational S^1 -homology groups for LM when M is the unit tangent bundle of an even dimensional sphere, and then apply this computation to answer the question left in [BFK15, Remark 9.6].

2 The involution on $A(M)$

In this section, we establish Theorem 2.1 which relates the involution τ_V on $A(M)$ defined in [Vog84, Vog85] to a geometric involution on $H_*^{S^1}(LM; \mathbb{Q}) = H_*(ES^1 \times_{S^1} LM; \mathbb{Q})$ where $ES^1 \times_{S^1} LM$ is the Borel construction and LM is the free loop space with S^1 -action given by $(z \cdot f)(w) = f(z \cdot w)$ for $z, w \in S^1 \subset \mathbb{C}$ and $f : S^1 \rightarrow M$. The geometric involution on $H_i^{S^1}(LM, *, \mathbb{Q})$ is induced by the involution

$$T : ES^1 \times_{S^1} LM \rightarrow ES^1 \times_{S^1} LM : [e, f] \mapsto [\bar{e}, \bar{f}]$$

where ES^1 is modelled here by the infinite-dimensional sphere $S^\infty = \bigcup_n S^{2n-1}$ which is naturally contained in $\mathbb{C}^\infty = \bigcup_n \mathbb{C}^n$. Also \bar{e} is the complex conjugate of $e \in ES^1 \subset \mathbb{C}^\infty$ and $\bar{f}(x) := f(\bar{x})$ for $x \in S^1$.

Let $*$ $\in M$ and let $H_i^{S^1}(LM, *, \mathbb{Q}) = H_i(ES^1 \times_{S^1} LM, ES^1 \times_{S^1} *, \mathbb{Q})$, then the involution T on the pair of spaces $(ES^1 \times_{S^1} LM, ES^1 \times_{S^1} *)$ induces an involution T_* on $H_i^{S^1}(LM, *, \mathbb{Q})$. Recall that $A(-)$ is a functor from the category of continuous maps of topological spaces to itself [Wal78]. Since the constant map $M \rightarrow *$ induces a retraction $A(M) \rightarrow A(*)$, then the inclusion $*$ $\rightarrow M$ induces the inclusion $A(*) \rightarrow A(M)$. Furthermore, since the involution τ_V is a natural transformation, it restricts to the involution on $A(*)$ and hence we have an involution τ_{V*} on $\pi_i(A(M), A(*))$. Burghlea [Bur86], Burghlea-Fiedorowicz [BF86], Goodwillie [Goo85, Goo86] and Waldhausen [Wal78] have proved that $\pi_{i+1}(A(M), A(*)) \otimes \mathbb{Q} \cong H_i^{S^1}(LM, *, \mathbb{Q})$ for all i (c.f. [BFK15, Corollary 7.11]). We can further obtain the following theorem.

Theorem 2.1. *For a simply-connected compact manifold M , the isomorphism*

$$\pi_{i+1}(A(M), A(*)) \otimes \mathbb{Q} \cong H_i^{S^1}(LM, *, \mathbb{Q})$$

can be chosen to be anti-equivariant with respect to the involutions τ_{V} and T_* . That is, there is an isomorphism $\pi_{i+1}(A(M), A(*)) \otimes \mathbb{Q} \rightarrow H_i^{S^1}(LM, *, \mathbb{Q})$ such that the following diagram commutes*

$$\begin{array}{ccc} \pi_{i+1}(A(M), A(*)) \otimes \mathbb{Q} & \longrightarrow & H_i^{S^1}(LM, *, \mathbb{Q}) \\ \tau_{V*} \downarrow & & \downarrow -T_* \\ \pi_{i+1}(A(M), A(*)) \otimes \mathbb{Q} & \longrightarrow & H_i^{S^1}(LM, *, \mathbb{Q}) \end{array}$$

The proof of Theorem 2.1 is given in the next subsections. We outline its proof here for the reader's convenience. Let X be a simply-connected simplicial set whose geometric realization $|X|$ is homeomorphic to M . Let $K_i(\mathbb{Z}[G(X)])$ be the i -th K-theory group of the simplicial group ring $\mathbb{Z}[G(X)]$ where $G(X)$ is the Kan loop group of X (see Section 2.1 and [GJ09, p.276]). Denote by $\tilde{K}_*(\mathbb{Z}[G(X)])$ the cokernel of the natural map $K_*(\mathbb{Z}) \rightarrow K_*(\mathbb{Z}[G(X)])$. Waldhausen [Wal78] has proved that there is an isomorphism

$$\pi_i(A(|X|), A(*)) \otimes \mathbb{Q} \cong \tilde{K}_i(\mathbb{Z}[G(X)]) \otimes \mathbb{Q} \quad (1)$$

and Burghlea [Bur86] and Goodwillie [Goo86] proved that

$$\tilde{K}_{i+1}(\mathbb{Z}[G(X)]) \otimes \mathbb{Q} \cong H_i^{S^1}(L|X|, *, \mathbb{Q}). \quad (2)$$

In [BF85], Burghlea and Fiedorowicz defined an involution τ_{BF*} on $\tilde{K}_*(\mathbb{Z}[G(X)])$ (see Section 2.1.2 for details). Our strategy to prove Theorem 2.1 is to show that the isomorphisms (1) and (2) can be chosen to be equivariant and anti-equivariant, respectively, with respect to the involutions τ_{V*} , τ_{BF*} and T_* .

2.1 The relation between involutions on $A(|X|)$ and $K_*(\mathbb{Z}[G(X)])$

In this section, let X be a connected simplicial set which is not necessarily simply-connected. We will introduce the involutions on $K(\mathbb{Z}[G(X)])$ defined by Burghlea-Fiedorowicz and Vogell, respectively, in Sections 2.1.2 and 2.1.3. In Section 2.1.4, we prove the two involutions coincide (c.f. Lemma 2.7) and deduce the following theorem.

Theorem 2.2. *Let X be a connected simplicial set. For each $i \geq 0$, there is an isomorphism $\pi_i(A(|X|), A(*)) \otimes \mathbb{Q} \rightarrow \tilde{K}_i(\mathbb{Z}[G(X)]) \otimes \mathbb{Q}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \pi_i(A(|X|), A(*)) \otimes \mathbb{Q} & \longrightarrow & \tilde{K}_i(\mathbb{Z}[G(X)]) \otimes \mathbb{Q} \\ \tau_{V*} \downarrow & & \downarrow \tau_{BF*} \\ \pi_i(A(|X|), A(*)) \otimes \mathbb{Q} & \longrightarrow & \tilde{K}_i(\mathbb{Z}[G(X)]) \otimes \mathbb{Q} \end{array}$$

2.1.1 Geometric realization of simplicial functors

For introducing the involutions defined by Vogell and Burghlea-Fiedorowicz, we recall from [HKV⁺02, Section 5.2] a general method to induce a map between topological spaces out of a (contravariant) functor of simplicial categories. Given a contravariant functor $F : C \rightarrow D$ between small categories, it induces a map $N.F$ from the nerve $N.C$ to $N.D$ where $N_n F : N_n C \rightarrow N_n D$ is given by $(g_1, \dots, g_n) \mapsto (F(g_n), \dots, F(g_1))$ for each $n \geq 1$ and composable morphisms g_i in the category C . Let $|C|$ denote the classifying space of the category C , i.e., the geometric realization of the nerve $N.C$. Then the map $N.F$ is anti-simplicial (i.e. $s_j \circ N_n F = N_{n+1} F \circ s_{n-j}$ and $d_j \circ N_n F = N_{n-1} F \circ d_{n-j}$ for $n \geq 1$ and $j \leq n$) and hence induces a map $|C| \rightarrow |D|$ via

$$\begin{array}{ccc} N_n C \times |\Delta^n| & \longrightarrow & N_n D \times |\Delta^n| \\ (x, s) & \mapsto & (N_n F(x), \Phi_n(s)) \end{array}$$

where $\Phi_n : |\Delta^n| \rightarrow |\Delta^n|$ is the simplicial homeomorphism of the geometric realization $|\Delta^n|$ of the standard simplex Δ^n which reverses the order of the vertices. More generally, let $F : C \rightarrow D$ be a simplicial contravariant functor of simplicial categories. As $F_n : C_n \rightarrow D_n$ is a contravariant functor of categories for each dimension n , repeating the previous construction in each dimension n gives rise to a map $N.F$ from the bisimplicial set $N.C$ to $N.D$, which is anti-simplicial in the first index and simplicial in the second index. Let $|C|$ denote the classifying space of the simplicial category C , namely, the double geometric realization of the nerve $N.C$. (The double geometric realization is homeomorphic to the geometric realization of the diagonal of the bisimplicial set, see [Qui73, p.94]). Then $N.F$ induces a map $|F| : |C| \rightarrow |D|$ via the map

$$\begin{array}{ccc} N_n C_k \times |\Delta^n| \times |\Delta^k| & \longrightarrow & N_n D_k \times |\Delta^n| \times |\Delta^k| \\ (x, s, t) & \mapsto & (N_n F_k(x), \Phi_n(s), t) \end{array}$$

In summary, we have the following lemma.

Lemma 2.3. *Every simplicial contravariant functor $F : C \rightarrow D$, of simplicial small categories induces a map $|F| : |C| \rightarrow |D|$ between the classifying spaces of the simplicial categories in a natural way. In particular, every anti-involution $C \rightarrow C$ (i.e. a simplicial contravariant functor whose square is the identity functor) induces an involution $|C| \rightarrow |C|$.*

2.1.2 Burghilea-Fiedorowicz's involution

Let us recall Burghilea and Fiedorowicz's involution τ_{BF} first. Denote the simplicial ring $\mathbb{Z}[G(X)]$ by R for short and follow Waldhausen[Wal78] to define $K(R)$. Consider $\pi_0 : R \rightarrow \pi_0(R)$ as a map of simplicial rings and define the simplicial monoid $\widehat{GL}_n(R)$ by the pull back diagram

$$\begin{array}{ccc} \widehat{GL}_n(R) & \longrightarrow & M_n(R) \\ \downarrow & & \downarrow \\ GL_n(\pi_0 R) & \longrightarrow & M_n(\pi_0 R) \end{array}$$

where $M_n(R)$ is the simplicial ring of the $n \times n$ matrices in R and the bottom horizontal map is the inclusion of the invertible matrices. Let $B\widehat{GL}(R)$ be the classifying space of the simplicial monoid $\widehat{GL}(R) = \varinjlim_n \widehat{GL}_n(R)$ and then apply Quillen's plus construction to define

$$K(R) := \mathbb{Z} \times B\widehat{GL}(R)^+$$

Since the simplicial monoid $\widehat{GL}(R)$ can be regarded as a simplicial category in a canonical way and a simplicial contravariant functor $\widehat{GL}(R) \rightarrow \widehat{GL}(R)$ can be given by

$$\left(\widehat{GL}_n(R)\right)_p \rightarrow \left(\widehat{GL}_n(R)\right)_p \subset M_n(R_p) : (a_{ij}) \mapsto (\overline{a_{ji}})$$

where the conjugation \overline{a} of $a \in R_p$ is induced by linearly extending the inverse map of the group $(G(X))_p$, then it follows from Lemma 2.3 that this induces an involution on the classifying space $B\widehat{GL}(R)$. Applying the plus construction, this gives the Burghilea-Fiedorowicz's involution τ_{BF} on $K(R)$ and hence induces the involution τ_{BF*} on $K_*(R) = \pi_* K(R)$.

Since the constant map $X \rightarrow *$ induces a retraction $K(\mathbb{Z}[G(X)]) \rightarrow K(\mathbb{Z})$, then the inclusion $* \rightarrow X$ induces a monomorphism $K_*(\mathbb{Z}[*]) \rightarrow K_*(\mathbb{Z}[G(X)])$ which commutes with the involution τ_{BF*} . This induces an involution on $\tilde{K}_*(\mathbb{Z}[G(X)]) = K_*(\mathbb{Z}[G(X)])/K_*(\mathbb{Z}[*])$ which is also denoted by τ_{BF*} .

2.1.3 Vogell's involution

Recall that Waldhausen defined in [Wal78, PROPOSITION 2.2] a linearization map $A(|X|) \rightarrow K(\mathbb{Z}[G(X)])$ which is a rational homotopy equivalence, and Vogell pointed out that there is an involution on $K(\mathbb{Z}[G(X)])$ which is compatible with the involution τ_V on $A(|X|)$ under this linearization map. In order to define this involution on $K(R) = K(\mathbb{Z}[G(X)])$, we need to recall another equivalent definition of $K(R)$ [Wal85, p.393]. Consider the category $\mathcal{M}(R)$ of simplicial (right) modules over R and their R -linear maps. Given two simplicial modules A and B in $\mathcal{M}(R)$, we say B is obtained from A by attaching of an n -cell if there is a pushout diagram

$$\begin{array}{ccc} R[\partial\Delta^n] & \longrightarrow & R[\Delta^n] \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

where $R[Y]$ denotes the simplicial R -module generated by the simplicial set Y , namely, for each n , $(R[Y])_n$ is the free R_n -module generated by Y_n . Define \mathcal{C} to be the full subcategory of the modules which are obtainable from the zero module by attaching of finitely many cells. Denote $R[\vee_k S^n]/R[*]$ by A_k^n where $S^n = \Delta^n/\partial\Delta^n$ and $\vee_k S^n$ is the one point union of k -copies of S^n at the point $*$. Let $\mathcal{C}_{A_k^n}$ denote the connected component of \mathcal{C} containing A_k^n . Taking direct limits with respect to the functors

$\mathcal{C}_{A_k^n} \rightarrow \mathcal{C}_{A_k^{n+1}}$ and $\mathcal{C}_{A_k^n} \rightarrow \mathcal{C}_{A_{k+1}^n}$ induced by the tensor product $\otimes_{\mathbb{Z}} \mathbb{Z}[S^1]/\mathbb{Z}[*]$ and the natural inclusion $\vee_k S^n \rightarrow \vee_{k+1} S^n$, one can define

$$K'(R) := \mathbb{Z} \times \left| \varinjlim_{n,k} \mathcal{C}_{A_k^n} \right|^+$$

which is homotopy equivalent to $K(R) = \mathbb{Z} \times \widehat{BGL}(R)^+ [\text{Wal85, p.393}]$. Vogell also mentioned that $K'(R)$ can be reconstructed from a larger category of R -modules by including duality data which leads to the desired involution. We now describe this construction in detail: First note that $R[S^n]/R[*]$ can be regarded as a right $R \otimes_{\mathbb{Z}} R$ -module by $r \cdot (s \otimes t) = \bar{t}rs$ for $r \in R_k$ and $s \otimes t \in R_k \otimes R_k$. Hence, if A and A' are simplicial modules in the category \mathcal{C} then any $R \otimes_{\mathbb{Z}} R$ -map $\omega : A \otimes_{\mathbb{Z}} A' \rightarrow R[S^n]/R[*]$ induces naturally a bilinear map

$$\pi_q(A \otimes_R \pi_0 R) \times \pi_{n-q}(A' \otimes_R \pi_0 R) \rightarrow \pi_n((A \otimes_R \pi_0 R) \otimes_{\mathbb{Z}} (A' \otimes_R \pi_0 R)) \rightarrow \pi_n(\pi_0 R[S^n]/\pi_0 R[*]) \cong \pi_0 R \quad (3)$$

for all $0 \leq q \leq n$. (For example $\pi_*(A_k^n \otimes_R \pi_0 R) = H_*((\vee_k S^n) \wedge G_+) \times^G E, * \times^G E; \pi_0 R$ where $G = G(X)$ and E is the universal G -bundle, c.f. [Vog85, p.285] and [Vog84, p.171].) In particular, if the bilinear maps (3) induce isomorphisms $\pi_q(A \otimes_R \pi_0 R) \cong \text{Hom}_{\pi_0 R}(\pi_{n-q}(A' \otimes_R \pi_0 R), \pi_0 R)$ for all $0 \leq q \leq n$, then we call ω a *linear n -duality map*.

Example 2.4. Denote $G(X)$ by G for short and let

$$(\vee_k S^n) \wedge G_+ \wedge (\vee_k S^m) \wedge G_+ \rightarrow S^{n+m} \wedge G_+$$

be the map induced by the smash product $S^n \wedge S^m \rightarrow S^{n+m}$ and the map $G \times G \rightarrow G, (g, h) \mapsto h^{-1}g$. By \mathbb{Z} -linearization $\mathbb{Z}[(\vee_k S^n) \wedge G_+ \wedge (\vee_k S^m) \wedge G_+] \rightarrow \mathbb{Z}[S^{n+m} \wedge G_+] = R[S^{n+m}]/R[*]$, this induces a linear $(n+m)$ -duality map

$$\mu_{n,m} : A_k^n \otimes_{\mathbb{Z}} A_k^m \rightarrow R[S^{n+m}]/R[*]$$

(c.f. [Vog85, p.285, Example 1.7.])

Example 2.5. Let $\omega : A \otimes_{\mathbb{Z}} A' \rightarrow R[S^n]/R[*]$ be a linear n -duality map and let $\bar{\omega}$ be the composition

$$A' \otimes_{\mathbb{Z}} A \cong A \otimes_{\mathbb{Z}} A' \xrightarrow{\omega} R[S^n]/R[*] \rightarrow R[S^n]/R[*]$$

where the latter map is induced by

$$\begin{array}{ccccc} S^n \wedge G_+ & \approx & \wedge^n S^1 \wedge G_+ & \longrightarrow & \wedge^n S^1 \wedge G_+ & \approx & S^n \wedge G_+ \\ & & [x_1, x_2, \dots, x_n, g] & \mapsto & [x_n, x_{n-1}, \dots, x_1, g^{-1}] & & \end{array}$$

Then $\bar{\omega}$ is a linear n -duality map (c.f. [Vog85, p.301])

Let \mathcal{DC}^n be the category in which an object is a triple (A, A', ω) where A and A' are simplicial modules in the category \mathcal{C} and ω is a linear n -duality map. A morphism in \mathcal{DC}^n is a pair

$$(f, f') : (A, A', \omega) \rightarrow (B, B', \eta)$$

where $f : A \rightarrow B$ and $f' : B' \rightarrow A'$ are morphisms in \mathcal{C} such that the following diagram commutes:

$$\begin{array}{ccc} A \otimes_{\mathbb{Z}} B' & \xrightarrow{f \otimes \text{id}} & B \otimes_{\mathbb{Z}} B' \\ \text{id} \otimes_{\mathbb{Z}} f' \downarrow & & \eta \downarrow \\ A \otimes_{\mathbb{Z}} A' & \xrightarrow{\omega} & R[S^n]/R[*] \end{array}$$

Define

$$\mathcal{DK}(R) := \mathbb{Z} \times \left| \varinjlim_{n,m,k} \mathcal{DC}_{(A_k^n, A_k^m, \mu_{n,m})}^{n+m} \right|^+$$

where $\mathcal{DC}_{(A_k^n, A_k^m, \mu_{n,m})}^{n+m}$ is the connected component of \mathcal{DC}^{n+m} containing the triple $(A_k^n, A_k^m, \mu_{n,m})$ and the limit here is taken with respect to the functors induced by the tensor product $\otimes_{\mathbb{Z}} \mathbb{Z}[S^1]/\mathbb{Z}[*]$ and the

natural inclusion $\vee_k S^n \rightarrow \vee_{k+1} S^n$. By Lemma 2.3, an involution $\bar{\tau}_V$ on $\mathcal{DK}(R)$ can be induced by the anti-involution (contravariant functor) $\mathcal{DC}^{n+m} \rightarrow \mathcal{DC}^{n+m}$ which sends an object (A, A', ω) to $(A', A, \bar{\omega})$ and sends a morphism (f, f') to (f', f) . Completely analogous to the proof of [Vog85, Corollary 1.16], one can show that there is a homotopy equivalence

$$\mathcal{DK}(R) \rightarrow K'(R) \quad (4)$$

induced by $\mathcal{DC}_{(A_k^n, A_k^m, \mu_{n,m})}^{n+m} \rightarrow \mathcal{C}_{A_k^n}$ which maps an object (A, A', ω) to A and maps a morphism (f, f') to f . This leads to an involution on $K'(R)$ and hence on $K(R)$, as $K(R)$ is homotopy equivalent to $K'(R)$ (c.f. [Wal85, 396. Corollary]).

Note that the constant map $X \rightarrow *$ induces a retraction $\mathcal{DK}(\mathbb{Z}[G(X)]) \rightarrow \mathcal{DK}(\mathbb{Z}[*])$ and the inclusion $\mathcal{DK}(\mathbb{Z}[*]) \rightarrow \mathcal{DK}(\mathbb{Z}[G(X)])$ commutes with the involution $\bar{\tau}_V$. This induces the involution $\bar{\tau}_{V*}$ on $\pi_*(\mathcal{DK}(\mathbb{Z}[G(X)]), \mathcal{DK}(\mathbb{Z}[*]))$.

Lemma 2.6. *Let $A(|X|)$ denote the Waldhausen's K -theory of the geometric realization of X . Then for each i there is a map $A(|X|) \rightarrow \mathcal{DK}(\mathbb{Z}[G(X)])$ inducing an isomorphism*

$$\pi_i(A(|X|), A(*)) \otimes \mathbb{Q} \rightarrow \pi_i(\mathcal{DK}(\mathbb{Z}[G(X)]), \mathcal{DK}(\mathbb{Z}[*])) \otimes \mathbb{Q}$$

such that the following diagram commutes:

$$\begin{array}{ccc} \pi_i(A(|X|), A(*)) \otimes \mathbb{Q} & \longrightarrow & \pi_i(\mathcal{DK}(\mathbb{Z}[G(X)]), \mathcal{DK}(\mathbb{Z}[*])) \otimes \mathbb{Q} \\ \tau_{V*} \downarrow & & \downarrow \bar{\tau}_{V*} \\ \pi_i(A(|X|), A(*)) \otimes \mathbb{Q} & \longrightarrow & \pi_i(\mathcal{DK}(\mathbb{Z}[G(X)]), \mathcal{DK}(\mathbb{Z}[*])) \otimes \mathbb{Q} \end{array}$$

Proof. In [Wal78, p.42, COROLLARY AND/OR DEFINITION], Waldhausen defines $A(|X|)$ by certain category whose objects are free pointed simplicial (right) $G(X)$ -sets Y . He also gives a linearization map $A(|X|) \rightarrow K'(\mathbb{Z}[G(X)])$ by sending a free pointed simplicial $G(X)$ -set to the free simplicial $\mathbb{Z}[G(X)]$ -module generated by the nonbasepoint elements. This linearization map fits into following commutative diagram

$$\begin{array}{ccc} \mathcal{DK}(|X|) & \longrightarrow & \mathcal{DK}(\mathbb{Z}[G(X)]) \\ \simeq \downarrow & & \downarrow \simeq \\ A(|X|) & \longrightarrow & K'(\mathbb{Z}[G(X)]) \end{array} \quad (5)$$

where the right vertical map is the map (4) and completely analogous to the construction of $\mathcal{DK}(R)$, the topological space $\mathcal{DK}(|X|)$ (denoted by $\mathbb{Z} \times |\varinjlim_{k,l,m} h\mathcal{DU}_k^{l,m}(G)|^+$ in [Vog85]) is constructed from a

category of free pointed simplicial $G(X)$ -sets by including some duality data. In the diagram (5) the left vertical map and the top horizontal map are defined analogously to the right vertical map and the bottom horizontal map, respectively. Both vertical maps are homotopy equivalences by [Vog85, Corollary 1.16]. Since the bottom map is a rational homotopy equivalence [Wal78, PROPOSITION 2.2], so is the top map. Analogous to the involution on $\mathcal{DK}(\mathbb{Z}[G(X)])$, an involution on $\mathcal{DK}(|X|)$ is defined by Vogell [Vog85, p.301] such that the top map is equivariant, and this involution induces the involution τ_V on $A(|X|)$ through the left vertical homotopy equivalence in the diagram (5). Moreover the diagram restricts to a commutative diagram when $X = *$. Then a homotopy inverse of the left vertical map in the diagram (5) composed with the top horizontal map gives the desired map $A(|X|) \rightarrow \mathcal{DK}(\mathbb{Z}[G(X)])$ and this completes the proof. \square

2.1.4 The involutions τ_{BF} and $\bar{\tau}_V$ coincide

We are going to show the two involutions τ_{BF} and $\bar{\tau}_V$ coincide, namely,

Theorem 2.7. *For each simplicial set X , there is a homotopy equivalence*

$$\eta_X : K(\mathbb{Z}[G(X)]) \rightarrow \mathcal{DK}(\mathbb{Z}[G(X)])$$

such that the first diagram below commutes up to homotopy and the second diagram commutes

$$\begin{array}{ccc}
K(\mathbb{Z}[G(X)]) & \xrightarrow{\eta_X} & \mathcal{D}K(\mathbb{Z}[G(X)]) \\
\tau_{BF} \downarrow & & \downarrow \bar{\tau}_V \\
K(\mathbb{Z}[G(X)]) & \xrightarrow{\eta_X} & \mathcal{D}K(\mathbb{Z}[G(X)])
\end{array}
\quad
\begin{array}{ccc}
K(\mathbb{Z}[*]) & \xrightarrow{\eta_*} & \mathcal{D}K(\mathbb{Z}[*]) \\
\downarrow & & \downarrow \\
K(\mathbb{Z}[G(X)]) & \xrightarrow{\eta_X} & \mathcal{D}K(\mathbb{Z}[G(X)])
\end{array}
\quad (6)$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc}
& & \mathcal{DC}^0_{\cdot(A_k^0, A_k^0, \mu_{0,0})} & \xrightarrow{\mathcal{D}\Sigma} & \mathcal{DC}^{n+m}_{\cdot(A_k^n, A_k^m, \mu_{n,m})} & \xleftarrow{\mathcal{D}\Upsilon} & \mathcal{DC}^{n+m}_{(A_k^n, A_k^m, \mu_{n,m})} & \\
& \nearrow \Phi & \downarrow & & \downarrow & & \downarrow & \\
\widehat{GL}_k(R) & & & & & & & \\
& \searrow \Psi & \downarrow & \xrightarrow{\Sigma} & \downarrow & \xleftarrow{\Upsilon} & \downarrow & \\
& & \mathcal{C}_{\cdot A_k^0} & & \mathcal{C}_{\cdot A_k^n} & & \mathcal{C}_{A_k^n} &
\end{array}
\quad (7)$$

where the notations are defined below.

1. \mathcal{DC}^n is a simplicial category defined as follow: for each p , the objects in the category \mathcal{DC}_p^n are the same with those in \mathcal{DC}^n , and a morphism $(A, A', \omega) \rightarrow (B, B', \eta)$ in \mathcal{DC}_p^n consists of a pair of morphisms $F : A \otimes_{\mathbb{Z}} \mathbb{Z}[\Delta^p] \rightarrow B$ and $F' : B' \otimes_{\mathbb{Z}} \mathbb{Z}[\Delta^p] \rightarrow A'$ in the category \mathcal{C} such that the following diagram commutes:

$$\begin{array}{ccccc}
A \otimes_{\mathbb{Z}} \mathbb{Z}[\Delta^p] \otimes_{\mathbb{Z}} B' & \xrightarrow{\sim} & A \otimes_{\mathbb{Z}} B' \otimes_{\mathbb{Z}} \mathbb{Z}[\Delta^p] & \xrightarrow{id \otimes_{\mathbb{Z}} F'} & A \otimes_{\mathbb{Z}} A' \\
F \otimes_{\mathbb{Z}} id \downarrow & & & & \downarrow \omega \\
B \otimes_{\mathbb{Z}} B' & \xrightarrow{\eta} & & & R[S^n]/R[*]
\end{array}$$

The canonical anti-involution in \mathcal{DC}^n is defined similarly as that in the category \mathcal{DC}^n . Denote by $\mathcal{DC}^{n+m}_{\cdot(A_k^n, A_k^m, \mu_{n,m})}$ the connected component of \mathcal{DC}^{n+m} containing $(A_k^n, A_k^m, \mu_{n,m})$ and regard $\mathcal{DC}^{n+m}_{(A_k^n, A_k^m, \mu_{n,m})}$ as a constant simplicial category. The horizontal map $\mathcal{D}\Upsilon : \mathcal{DC}^{n+m}_{(A_k^n, A_k^m, \mu_{n,m})} \rightarrow \mathcal{DC}^{n+m}_{\cdot(A_k^n, A_k^m, \mu_{n,m})}$ is induced by associating $f : A \rightarrow B$ to $f \otimes_{\mathbb{Z}} c : A \otimes_{\mathbb{Z}} \mathbb{Z}[\Delta^p] \rightarrow B \otimes_{\mathbb{Z}} \mathbb{Z}$, where $c : \mathbb{Z}[\Delta^p] \rightarrow \mathbb{Z}$ is the \mathbb{Z} -linear map induced by the constant map $\Delta^p \rightarrow *$.

2. The horizontal map $\Upsilon : \mathcal{C}_{A_k^n} \rightarrow \mathcal{C}_{\cdot A_k^n}$ is defined similarly to Point 1, where \mathcal{C}_{\cdot} is the simplicial category which is the blow up of \mathcal{C} and $\mathcal{C}_{A_k^n}$ denotes the connected component of \mathcal{C}_{\cdot} containing A_k^n (c.f. [Wal85, p.396]).
3. The three vertical maps are all induced by the projections onto the first factors.
4. The horizontal maps $\mathcal{D}\Sigma : \mathcal{DC}^0_{\cdot(A_k^0, A_k^0, \mu_{0,0})} \rightarrow \mathcal{DC}^{n+m}_{\cdot(A_k^n, A_k^m, \mu_{n,m})}$ and $\Sigma : \mathcal{C}_{\cdot A_k^0} \rightarrow \mathcal{C}_{\cdot A_k^n}$ are induced from the tensor product $\otimes_{\mathbb{Z}} \mathbb{Z}[S^n]/\mathbb{Z}[*]$ and $\otimes_{\mathbb{Z}} \mathbb{Z}[S^m]/\mathbb{Z}[*]$, respectively.
5. Note each $M \in \left(\widehat{GL}_k(R) \right)_p \subset M_k(R_p)$ determines a unique morphism $f_M : A_k^0 \otimes_{\mathbb{Z}} \mathbb{Z}[\Delta^p] \rightarrow A_k^0$ in the category \mathcal{C} such that $(f_M)_p(v \otimes \iota_p) = Mv$ for any column vector $v \in R_p^k = (A_k^0)_p$ and the non-degenerate p -simplex ι_p of Δ^p . This induces the natural map $\Psi : \widehat{GL}_k(R) \rightarrow \mathcal{C}_{\cdot A_k^0}$ when consider $\widehat{GL}_k(R)$ as a simplicial category in the usual way. Similarly, $\Phi : \widehat{GL}_k(R) \rightarrow \mathcal{DC}^0_{\cdot(A_k^0, A_k^0, \mu_{0,0})}$ is the natural map which maps each $M \in \left(\widehat{GL}_k(R) \right)_p$ to the morphism $(f_M, f_{\overline{M}^T}) : (A_k^0, A_k^0, \mu_{0,0}) \rightarrow (A_k^0, A_k^0, \mu_{0,0})$.

In the diagram (7), since the simplicial maps Φ , $\mathcal{D}\Sigma$ and $\mathcal{D}\Upsilon$ preserve the anti-involutions, and $\mathcal{D}\Upsilon$ is a homotopy equivalence, by passing to limits the composition of $\mathcal{D}\Sigma \circ \Phi$ with a homotopy inverse of $\mathcal{D}\Upsilon$ gives rise to a map

$$\eta_X : K(\mathbb{Z}[G(X)]) \rightarrow \mathcal{D}K(\mathbb{Z}[G(X)])$$

which fits into the commutative diagrams (6). To prove the theorem it suffices to prove η_X is a homotopy equivalence. Since the vertical map on the right in the diagram (7) is a homotopy equivalence when passing to limits, and the simplicial maps Ψ and Υ are both homotopy equivalences (c.f. [Wal85, p.396, Proposition 2.3.5]), by the commutativity of the diagram (7) we only need to show that the simplicial map Σ is a homotopy equivalence. For this, consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}_{\cdot, A_k^0} & \xrightarrow{\Sigma} & \mathcal{C}_{\cdot, A_k^n} \\ \uparrow & & \uparrow \\ \mathcal{C}_{\cdot}(A_k^0) & \longrightarrow & \mathcal{C}_{\cdot}(A_k^n) \end{array}$$

where $\mathcal{C}_{\cdot}(A_k^n)$ denotes the simplicial monoid of simplicial R -linear self-homotopy equivalences of A_k^n (c.f. [GJ09, I.7]), the vertical maps are the natural inclusions, and the bottom horizontal map is the restriction of the “suspension” map $\text{Hom}_R(A_k^0, A_k^0) \rightarrow \text{Hom}_R(A_k^n, A_k^n)$ induced from the tensor product $\mathbb{Z}[S^n]/\mathbb{Z}[*] \otimes_{\mathbb{Z}}$, where $\text{Hom}_R(A_k^n, A_k^n)$ is the simplicial monoid of simplicial R -linear self-maps of A_k^n . Since the vertical maps are homotopy equivalences [Wal85, p.396, Prop. 2.3.5], so it remains to show

$$\text{Hom}_R(A_k^0, A_k^0) \rightarrow \text{Hom}_R(A_k^n, A_k^n) \quad (8)$$

is a homotopy equivalence. To see this, let A be a simplicial abelian group and $\tilde{\mathbb{Z}}[S^n] = \mathbb{Z}[S^n]/\mathbb{Z}[*]$. As $A_p = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[\Delta^p], A)$ for each dimension p , define a map

$$A \xrightarrow{\tilde{\mathbb{Z}}[S^1] \otimes_{\mathbb{Z}}} \text{Hom}_{\mathbb{Z}}(\tilde{\mathbb{Z}}[S^1], \tilde{\mathbb{Z}}[S^1] \otimes A) \quad (9)$$

by sending a simplicial \mathbb{Z} -linear map $f : \mathbb{Z}[\Delta^p] \rightarrow A$ to the map $id \otimes f : \tilde{\mathbb{Z}}[S^1] \otimes \mathbb{Z}[\Delta^p] \rightarrow \tilde{\mathbb{Z}}[S^1] \otimes A$ in each dimension p .

Claim The map (9) is a homotopy equivalence.

Assuming this claim we prove the “suspension” map (8) is a homotopy equivalence. Since

$$\text{Hom}_R(A_k^n, A_k^n) = \bigoplus_{k \times k} \text{Hom}_R(A_1^n, A_1^n),$$

it suffices to prove that the “suspension” map (8) is a homotopy equivalence for $k = 1$. Firstly, it is true for $n = 1$ by using the claim above. Now assume it is true for $n = l - 1$. Then the fact that $\text{Hom}_R(A_1^0, A_1^0) \rightarrow \text{Hom}_R(A_1^l, A_1^l)$ is a homotopy equivalence just follows from that it is the composition of the homotopy equivalences

$$\text{Hom}_R(A_1^0, A_1^0) \xrightarrow{\tilde{\mathbb{Z}}[S^{l-1}] \otimes} \text{Hom}_R(A_1^{l-1}, A_1^{l-1}) \longrightarrow \text{Hom}_R(A_1^{l-1}, \text{Hom}_{\mathbb{Z}}(\tilde{\mathbb{Z}}[S^1], \tilde{\mathbb{Z}}[S^1] \otimes A_1^{l-1})) \cong \text{Hom}_R(A_1^l, A_1^l)$$

where the second map is induced by the map in the claim above when $A = A_1^{l-1}$. Then it follows that the “suspension” map (8) is a homotopy equivalence for all $k \geq 1$.

It remains to prove the claim above. The proof follows from specifying the arguments in [Wal85, p.396, Proposition 2.3.5] and a simplicial version of the proof for [Hat02, Proposition 4.66]. Let

$$c : \Delta^1 \times (A[\Delta^1] / A[\Delta^0]) \rightarrow A[\Delta^1] / A[\Delta^0]$$

be the contraction which linearly extends the standard contraction $\Delta^1 \times \Delta^1 \rightarrow \Delta^1$, whose restrictions to $\Delta^1 \times 0$ and $0 \times \Delta^1$ are the projections of Δ^1 onto the vertex 0 and whose restriction to $\Delta^1 \times 1$ is the identity map of Δ^1 . Let $P(A[\Delta^1] / A[\partial\Delta^1])$ (resp. $\Omega(A[\Delta^1] / A[\partial\Delta^1])$) denote the path space (resp.

the loop space) of $A[\Delta^1]/A[\partial\Delta^1]$ (see [GJ09, p.30]). Then the contraction c induces naturally a map $A[\Delta^1]/A[\Delta^0] \rightarrow P(A[\Delta^1]/A[\partial\Delta^1])$ which fits into the following commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{\quad\quad\quad} & A[\Delta^1]/A[\Delta^0] & \longrightarrow & A[\Delta^1]/A[\partial\Delta^1] \\
\downarrow \tilde{\mathbb{Z}}[S^1] \otimes & & \downarrow & & \downarrow id \\
Hom_{\mathbb{Z}}(\tilde{\mathbb{Z}}[S^1], \tilde{\mathbb{Z}}[S^1] \otimes A) & \xrightarrow{\sim} & \Omega(A[\Delta^1]/A[\partial\Delta^1]) & \longrightarrow & P(A[\Delta^1]/A[\partial\Delta^1]) & \longrightarrow & A[\Delta^1]/A[\partial\Delta^1]
\end{array}$$

where the horizontal maps are the natural fibrations. Since $A[\Delta^1]/A[\Delta^0]$ and $P(A[\Delta^1]/A[\partial\Delta^1])$ are contractible, the five lemma implies that the left vertical map is a weak homotopy equivalence. As the simplicial abelian groups A and $Hom_{\mathbb{Z}}(\tilde{\mathbb{Z}}[S^1], \tilde{\mathbb{Z}}[S^1] \otimes A)$ are Kan complexes, then the claim follows. This completes the proof of the theorem. \square

Proof of Theorem 2.2. Theorem 2.7 implies that the homotopy equivalence $\eta_X : K(\mathbb{Z}[G(X)]) \rightarrow \mathcal{DK}(\mathbb{Z}[G(X)])$ induces isomorphisms

$$\eta_{X*} : \tilde{K}_i(\mathbb{Z}[G(X)]) \xrightarrow{\cong} \pi_i(\mathcal{DK}(\mathbb{Z}[G(X)]), \mathcal{DK}(\mathbb{Z}))$$

such that the following diagram commutes

$$\begin{array}{ccc}
\tilde{K}_i(\mathbb{Z}[G(X)]) & \xrightarrow{\eta_{X*}} & \pi_i(\mathcal{DK}(\mathbb{Z}[G(X)]), \mathcal{DK}(\mathbb{Z})) \\
\downarrow \tau_{BF*} & & \downarrow \bar{\tau}_{V*} \\
\tilde{K}_i(\mathbb{Z}[G(X)]) & \xrightarrow{\eta_{X*}} & \pi_i(\mathcal{DK}(\mathbb{Z}[G(X)]), \mathcal{DK}(\mathbb{Z}))
\end{array}$$

for all i . This, together with Lemma 2.6, completes the proof. \square

2.2 The relation between involutions on $K_*(\mathbb{Z}[G(X)])$ and $H_i^{S^1}(L|X|)$

Let X be a simply-connected simplicial set and let $HD_i(X)$ denote the i -th dihedral homology for the simplicial Hermitian ring $\mathbb{Z}[G(X)]$ (c.f. [Lod96, p.193]). An anti-equivariant isomorphism $\tilde{K}_{i+1}(\mathbb{Z}[G(X)]) \otimes \mathbb{Q} \cong H_i^{S^1}(L|X|, *, \mathbb{Q})$ with respect to the involution τ_{BF*} and the geometric involution T_* , can be obtained by recalling the following results: On one hand, applying the results of [Lod96, p.195] directly to the Hermitian simplicial ring homomorphism $\mathbb{Z}[G(X)] \rightarrow \mathbb{Z}$ induced by $G(X) \rightarrow *$, one can see there is an isomorphism

$$Inv^- \tilde{K}_{i+1}(\mathbb{Z}[G(X)]) \otimes \mathbb{Q} \cong \widetilde{HD}_i(X) \otimes \mathbb{Q} := HD_i(X) \otimes \mathbb{Q} / HD_i(*) \otimes \mathbb{Q}.$$

On the other hand, [Dun89] proved that $HD_i(X) \otimes \mathbb{Q}$ is isomorphic to $H_i^{O(2)}(L|X|; \mathbb{Q})$ and a modification of the proof of [Lod90, 3.3.3 THEOREM] can also show this. (Note that a straightforward argument shows that the definitions of the dihedral homology groups in [Lod90] and [Lod96] are equivalent.) These facts, together with the isomorphism

$$H_i^{O(2)}(L|X|; \mathbb{Q}) \cong Inv^+ H_i^{S^1}(L|X|; \mathbb{Q}),$$

imply that

$$Inv^- \tilde{K}_{i+1}(\mathbb{Z}[G(X)]) \otimes \mathbb{Q} \cong Inv^+ H_i^{S^1}(L|X|, *, \mathbb{Q}).$$

Note that $\dim \tilde{K}_i(\mathbb{Z}[G(X)]) \otimes \mathbb{Q}$ is finite for each i if the geometric realization $|X|$ is a simply-connected compact manifold (c.f. [Dwy80, Wal78]), and since $\tilde{K}_{i+1}(\mathbb{Z}[G(X)]) \otimes \mathbb{Q} \cong H_i^{S^1}(L|X|, *, \mathbb{Q})$ (c.f. [Bur86, BF86, Goo85, Goo86]), then this implies the following theorem.

Theorem 2.8. *Let X be a simply-connected simplicial set such that the geometric realization $|X|$ is a compact manifold. Then there is an isomorphism $\tilde{K}_{i+1}(\mathbb{Z}[G(X)]) \otimes \mathbb{Q} \xrightarrow{\cong} H_i^{S^1}(L|X|, *, \mathbb{Q})$ for all i*

such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{K}_{i+1}(\mathbb{Z}[G(X)]) \otimes \mathbb{Q} & \xrightarrow{\cong} & H_i^{S^1}(L|X|, *, \mathbb{Q}) \\ \tau_{BF*} \downarrow & & \downarrow -T_* \\ \tilde{K}_{i+1}(\mathbb{Z}[G(X)]) \otimes \mathbb{Q} & \xrightarrow{\cong} & H_i^{S^1}(L|X|, *, \mathbb{Q}) \end{array}$$

Proof of Theorem 2.1. It follows immediately from Theorems 2.2 and 2.8. □

3 The involution on the pseudoisotopy space

Throughout this section let M be a compact smooth manifold, possibly with boundary. The goal of this section is to introduce the involution on pseudoisotopy spaces and to prove Lemma 3.1, which relates the involution on the pseudoisotopy space of M to the involution τ_V on $A(M)$. The proof is based on Waldhausen's manifold approach for $A(M)$ (c.f. [Wal82]) and some arguments of [Vog85]. We prove Theorem 1.1 at the end of this section.

A pseudoisotopy of M is defined to be a diffeomorphism of $M \times [0, 1]$ which is the identity on $M \times 0 \cup \partial M \times [0, 1]$. Then the pseudoisotopy space is the group of all such diffeomorphisms equipped with the smooth topology, i.e.,

$$P(M) := \text{Diff}(M \times [0, 1], \text{rel } M \times 0 \cup \partial M \times [0, 1]).$$

By Igusa's stabilization theorem [Igu88][Igu02, Theorem 6.2.2], the natural stabilization map

$$\Sigma : P(M) \rightarrow P(M \times [0, 1])$$

is k -connected if $\dim M \geq \max\{2k + 7, 3k + 4\}$. Then the stabilization

$$P(M) \rightarrow P(M \times [0, 1]) \rightarrow P(M \times [0, 1]^2) \rightarrow \dots$$

is eventually an isomorphism on homotopy groups. So define the stable pseudoisotopy space as the direct limit

$$\mathcal{P}(M) := \varinjlim_m P(M \times [0, 1]^m).$$

Following [Vog85, p.296], there is a canonical involution $\iota : P(M) \rightarrow P(M)$ given by

$$\iota(f) = (id_M \times r) \circ f \circ (id_M \times r) \circ ((f|_{M \times 1})^{-1} \times id_{[0, 1]})$$

where $r : [0, 1] \rightarrow [0, 1]$ is the reflection given by $r(t) = 1 - t$. Since the stabilization induces a map $\pi_* P(M) \rightarrow \pi_* P(M \times [0, 1])$ which anti-commutes with the canonical involution (c.f. [Hat78] or [Igu02, Proposition 6.2.1, Lemma 6.5.1(2)]), then define the involution ι_*^S on $\pi_* \mathcal{P}(M)$ to be the one compatible with the involution $(-1)^{\dim M} \iota_*$ on $\pi_* P(M)$ (c.f. [Igu02]), namely the following diagram commutes:

$$\begin{array}{ccc} \pi_k P(M) & \longrightarrow & \pi_k \mathcal{P}(M) \\ (-1)^{\dim M} \iota_* \downarrow & & \downarrow \iota_*^S \\ \pi_k P(M) & \longrightarrow & \pi_k \mathcal{P}(M) \end{array}$$

Waldhausen proved that for all $i \geq 0$ there is a short exact sequence

$$0 \rightarrow \pi_{i+2}^s(M_+) \rightarrow \pi_{i+2} A(M) \rightarrow \pi_i \mathcal{P}(M) \rightarrow 0$$

(c.f. [Wal82] and [WJR13]), where $\pi_{i+2}^s(M_+)$ denotes the $(i+2)$ -th stable homotopy group of the disjoint union of M with a point. We establishing the following lemma:

Lemma 3.1. *For each i there is a short exact sequence*

$$0 \rightarrow \pi_{i+2}^s(M_+) \xrightarrow{\alpha} \pi_{i+2}A(M) \xrightarrow{\beta} \pi_i\mathcal{P}(M) \rightarrow 0,$$

such that the following two conditions are satisfied :

- (1) *the image of the homomorphism α is contained in $\text{Inv}_{\tau_V^*}^+ \pi_{i+2}A(M)$.*
- (2) *the homomorphism β makes the following diagram commute:*

$$\begin{array}{ccc} \pi_{i+2}A(M) & \xrightarrow{\beta} & \pi_i\mathcal{P}(M) \\ -\tau_{V*} \downarrow & & \downarrow \iota_*^S \\ \pi_{i+2}A(M) & \xrightarrow{\beta} & \pi_i\mathcal{P}(M) \end{array}$$

In order to prove this lemma, we review Waldhausen's manifold approach to $A(M)$ (c.f. [Wal82] and [WJR13]).

Given a compact manifold M with boundary ∂M , a partition is a triple (M_0, F, M_1) of manifolds, as shown in Figure 1, where M_0 is a codimension 0 submanifold of $M \times I$ with $I = [a, b]$, M_1 is the closure

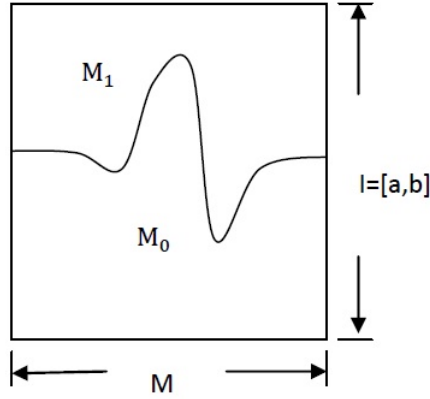


Figure 1: A partition (M_0, F, M_1)

of the complement of M_0 and F is the intersection $M_0 \cap M_1$. Furthermore, the frontier F is required to be disjoint from the bottom $M \times a$ and the top $M \times b$ and to be standard near $\partial M \times I$ in the sense that there exists a neighborhood W of $\partial M \times I$ satisfying that $W \cap F$ is equal to $W \cap M \times \{t\}$ for some $t \in I$. Let $D(M)$ be the simplicial set whose k -simplices are locally trivial families of partitions (M_0, F, M_1) (parameterized by the standard simplex Δ^k) and let $\underline{D}(M)$ be the simplicial subset of $D(M)$ of those partitions (M_0, F, M_1) satisfying that $p[\partial F]$ is the minimum of $p[F]$ where $p : M \times I \rightarrow I$ is the projection onto the second factor.

A partial ordering can be defined by letting $(M_0, F, M_1) < (M'_0, F', M'_1)$ if $M_0 \subset M'_0$ and the two inclusion maps

$$F \rightarrow M'_0 - (M_0 - F) \leftarrow F'$$

are homotopy equivalences. This partial ordering on $D(M)$ (resp. $\underline{D}(M)$) defines a simplicial partially ordered set, and hence a simplicial category which is denoted by $hD(M)$ (resp. $h\underline{D}(M)$). Let $hD_k^{l,m}(M)$ (resp. $h\underline{D}_k^{l,m}(M)$) be the connected component of $hD(M)$ (resp. $h\underline{D}(M)$) containing the particular partition given by attaching k m -handles trivially to $M \times [a, a']$ on $M \times a'$ in such a way that the complementary l -handles are trivially attached to $M \times [b', b]$ on $M \times b'$, $a < a' < b' < b$.

Let $d = \dim M$ and let J be a closed interval and consider the limit

$$\varinjlim_{k,l,m} hD_k^{l,m}(M \times J^{l+m-d})$$

where the maps in the direct system are given by the lower stabilization map

$$\sigma_l : h\underline{D}_k^{l,m}(M \times J^{l+m-d}) \rightarrow h\underline{D}_k^{l+1,m}(M \times J^{l+m+1-d}) \quad (10)$$

and the upper stabilization map

$$\sigma_u : h\underline{D}_k^{l,m}(M \times J^{l+m-d}) \rightarrow h\underline{D}_k^{l,m+1}(M \times J^{l+m+1-d}) \quad (11)$$

which are defined in [Vog85, p.298].

By a slight modification of the (anti-)involution on $hD(M)$ which is induced by turning a partition upside down, [Vog85, p.297] obtains an involution \mathcal{T} (up to homotopy) on $h\underline{D}(M)$ which satisfies $\sigma_l \circ \mathcal{T} \simeq \mathcal{T} \circ \sigma_u$ and $\sigma_u \circ \mathcal{T} \simeq \mathcal{T} \circ \sigma_l$, and hence \mathcal{T} induces an involution

$$\varinjlim_{k,l,m} h\underline{D}_k^{l,m}(M \times J^{l+m-d}) \rightarrow \varinjlim_{k,l,m} h\underline{D}_k^{l,m}(M \times J^{l+m-d})$$

which is compatible with the restriction of \mathcal{T} on $h\underline{D}_k^{L,L}(M \times J^{2L-d})$ for any $L \gg 0$. Denote by $\underline{D}_k^{l,m}(M)$ (resp. $\underline{D}_k^{l,m}(M)$) the simplicial set of the objects of the simplicial category $h\underline{D}_k^{l,m}(M)$ (resp. $h\underline{D}_k^{l,m}(M)$). Then similar to the above, one also gets the involution on the following limits

$$\varinjlim_{k,l,m} \underline{D}_k^{l,m}(M \times J^{l+m-d}), \varinjlim_{l,m} \underline{D}_0^{l,m}(M \times J^{l+m-d}), \varinjlim_{k,l,m} \underline{D}_k^{l,m}(M \times J^{l+m-d}), \varinjlim_{l,m} \underline{D}_0^{l,m}(M \times J^{l+m-d}).$$

Let $|\cdot|^+$ denote the space obtained by performing the plus construction on the geometric realization $|\cdot|$. (Note that if S is a simplicial category, then $|S|$ denote the classifying space.) By [Wal82], the natural map (induced by the inclusion)

$$\left| \varinjlim_{k,l,m} \underline{D}_k^{l,m}(M \times J^{l+m-d}) \right|^+ \rightarrow \left| \varinjlim_{k,l,m} h\underline{D}_k^{l,m}(M \times J^{l+m-d}) \right|^+ \quad (12)$$

has a homotopy fiber $\left| \varinjlim_{l,m} \underline{H}(M \times J^{l+m-d}) \right|$ where $\underline{H}(M \times J^{l+m-d}) = \underline{D}_0^{l,m}(M \times J^{l+m-d})$. Moreover, the map (12) is compatible with the involutions induced by \mathcal{T} (c.f. [Vog85, p.299]). Since there is a homotopy equivalence

$$A(M) \xrightarrow{\simeq} \left| \varinjlim_{k,l,m} h\underline{D}_k^{l,m}(M \times J^{l+m-d}) \right|^+$$

which makes the following diagram commute up to homotopy

$$\begin{array}{ccc} A(M) & \xrightarrow{\simeq} & \left| \varinjlim_{k,l,m} h\underline{D}_k^{l,m}(M \times J^{l+m-d}) \right|^+ \\ \tau_V \downarrow & & \downarrow \\ A(M) & \xrightarrow{\simeq} & \left| \varinjlim_{k,l,m} h\underline{D}_k^{l,m}(M \times J^{l+m-d}) \right|^+ \end{array}$$

where the right vertical map is the involution induced by \mathcal{T} (c.f. [Wal82][Vog85]), then the map (12) induces a long exact sequence

$$\cdots \longrightarrow \pi_{i+2} \left| \varinjlim_{k,l,m} \underline{D}_k^{l,m}(M \times J^{l+m-d}) \right|^+ \xrightarrow{\alpha'} \pi_{i+2} A(M) \xrightarrow{\gamma} \pi_{i+1} \left| \varinjlim_{l,m} \underline{H}(M \times J^{l+m-d}) \right| \longrightarrow \cdots \quad (13)$$

which is compatible with the involution τ_{V*} and the involutions induced by \mathcal{T} on $\pi_* \left| \varinjlim_{k,l,m} D_k^{l,m}(M \times J^{l+m-d}) \right|^+$ and $\pi_* \left| \varinjlim_{l,m} H(M \times J^{l+m-d}) \right|$. Moreover, [Wal82, p.153] proved that the sequence (13) splits, i.e., there is a homomorphism

$$q' : \pi_{i+2}A(X) \longrightarrow \pi_{i+2} \left| \varinjlim_{k,l,m} D_k^{l,m}(M \times J^{l+m-d}) \right|^+$$

such that $q' \circ \alpha' = id$.

Proof of Lemma 3.1. We first obtain the desired short exact sequence from the sequence (13). The argument on [Wal82, p.157] gives an explicit map

$$\theta : \left| \varinjlim_{k,l,m} D_k^{l,m}(M \times J^{l+m-d}) \right| \xrightarrow{\simeq} \Omega^\infty \Sigma^\infty(M_+)$$

which is a homotopy equivalence by [Wal87b, p.5] and [Wal87a]. Since $\pi_1 \left| \varinjlim_{k,l,m} D_k^{l,m}(M \times J^{l+m-d}) \right| \cong \pi_1^s(M_+)$ is an abelian group, then

$$\left| \varinjlim_{k,l,m} D_k^{l,m}(M \times J^{l+m-d}) \right|^+ = \left| \varinjlim_{k,l,m} D_k^{l,m}(M \times J^{l+m-d}) \right|.$$

Let

$$\alpha = \alpha' \circ \theta_*^{-1} : \pi_{i+2}^s(M_+) \rightarrow \pi_{i+2}A(M).$$

On the other hand, let $H(M \times J^{l+m-d})$ denote $\left| D_0^{l,m}(M \times J^{l+m-d}) \right|$. The proof of [Vog85, p.297, Proposition 2.2] shows that there is a homotopy equivalence $\Omega H(M \times J^{2L-d}) \xrightarrow{\simeq} P(M \times J^{2L-d})$ such that the induced isomorphisms

$$\pi_{i+1}H(M \times J^{2L-d}) \xrightarrow{\cong} \pi_i P(M \times J^{2L-d}) \quad (14)$$

for all i anti-commute with the involutions, namely, the following diagram commutes for all i :

$$\begin{array}{ccc} \pi_{i+1}H(M \times J^{2L-d}) & \xrightarrow{\cong} & \pi_i P(M \times J^{2L-d}) \\ \downarrow & & \downarrow -\tau_* \\ \pi_{i+1}H(M \times J^{2L-d}) & \xrightarrow{\cong} & \pi_i P(M \times J^{2L-d}) \end{array}$$

where the left vertical map is the involution induced by \mathcal{T} . Moreover, by Igusa's stabilization theorem [Igu88], when $L \gg 0$, there are isomorphisms

$$\pi_{i+1} \left| \varinjlim_{l,m} H(M \times J^{l+m-d}) \right| \xleftarrow{\cong} \pi_{i+1} \left| \underline{H}(M \times J^{2L-d}) \right| \xrightarrow{\cong} \pi_{i+1}H(M \times J^{2L-d}) \quad (15)$$

and

$$\pi_i P(M \times J^{2L-d}) \xrightarrow{\cong} \pi_i \mathcal{P}(M) \quad (16)$$

which are compatible with the involutions. The composition of the map γ in the sequence (13) and the isomorphisms (14)(15) and (16) give the homomorphism β which fits into a short exact sequence

$$0 \rightarrow \pi_{i+2}^s(M_+) \xrightarrow{\alpha} \pi_{i+2}A(M) \xrightarrow{\beta} \pi_i \mathcal{P}(M) \rightarrow 0.$$

This sequence splits as the sequence (13) splits. It is clear that β satisfies the condition (2) in the lemma.

It remains to show the condition (1), namely, the image of the homomorphism α is contained in $\text{Inv}_{\tau_{V*}}^+ \pi_{i+2} A(M)$. Since $\alpha = \alpha' \circ \theta_*^{-1}$ and α' is compatible with the involutions τ_{V*} and \mathcal{T} , it suffices to prove that the involution \mathcal{T} on $\varinjlim_{k,l,m} \underline{D}_k^{l,m}(M \times J^{l+m-d})$ is the identity up to homotopy. For this, we show that the following diagram commutes up to homotopy:

$$\begin{array}{ccc}
 \left| \varinjlim_{k,l,m} \underline{D}_k^{l,m}(M \times J^{l+m-d}) \right| & & \\
 \downarrow \mathcal{T} & \searrow \theta & \\
 & & \Omega^\infty \Sigma^\infty(M_+) \\
 & \nearrow \theta & \\
 \left| \varinjlim_{k,l,m} \underline{D}_k^{l,m}(M \times J^{l+m-d}) \right| & &
 \end{array} \tag{17}$$

We first show the diagram (17) commutes up to homotopy when M is a codimension 0 submanifold in \mathbb{R}^d . Consider the following diagram

$$\begin{array}{ccc}
 \underline{D}_k^{l,m}(M \times J^{l+m-d}) & & \\
 \downarrow \mathcal{T} & \searrow \Theta & \\
 & & \text{Map}(M \times J^{l+m-d} / \partial(M \times J^{l+m-d}), Q^{l+m}) \\
 & \nearrow \Theta & \\
 \underline{D}_k^{m,l}(M \times J^{l+m-d}) & &
 \end{array} \tag{18}$$

where Q^{l+m} denotes the space of germs of normally oriented $(l+m)$ -planes in \mathbb{R}^{l+m+1} and Θ is the map defined in [Wal82, p.156] which approximates the map θ . From the definition of Θ , it is easy to check that the diagram (18) commutes up to homotopy and it is compatible with the stabilizations (10), (11), and

$$\text{Map}(Y/\partial Y, Q^L) \rightarrow \text{Map}(Y \times J/\partial(Y \times J), Q^{L+1})$$

which is induced by the homeomorphism $Y \times J/\partial(Y \times J) \approx Y \wedge J/\partial J$ and the structural map $Q^L \wedge J/\partial J \rightarrow Q^{L+1}$, where $L = l + m$ and $Y = M \times J^{L-d}$ (c.f. [Wal82, pp.155-156]). Since

$$\varinjlim_L \text{Map}(M \times J^{L-d} / \partial(M \times J^{L-d}), Q^L) \simeq \Omega^\infty \Sigma^\infty(M_+)$$

and the map Θ approximates θ , then this implies the diagram (17) commutes up to homotopy and verifies the condition (1) when M is a codimension 0 submanifold in \mathbb{R}^d .

Now assume that M is not a codimension 0 submanifold in \mathbb{R}^d . By Whitney's embedding theorem, the manifold M can be embedded into \mathbb{R}^{d+N} for $N \gg 0$. Let $N(M)$ be a regular tubular neighborhood of M in \mathbb{R}^{d+N} . We have the following commutative diagram

$$\begin{array}{ccc}
 A^s(M) & \rightarrow & A(M) \\
 \downarrow & & \downarrow \eta \\
 A^s(N(M)) & \rightarrow & A(N(M))
 \end{array} \tag{19}$$

where

$$\begin{aligned}
 A^s(Y) &:= \varinjlim_{k,l,m} \underline{D}_k^{l,m}(Y \times J^{l+m-\dim Y}), \\
 A(Y) &:= \varinjlim_{k,l,m} h \underline{D}_k^{l,m}(Y \times J^{l+m-\dim Y})
 \end{aligned}$$

and the horizontal maps are induced from the natural inclusions and the vertical maps are essentially defined by

$$(M, F, N) \mapsto (p^{-1}[M], p^{-1}[F], p^{-1}[N])$$

(with a technical modification, c.f. [Wal82, p.175]) where $p = \pi \times id_I : N(X) \times I \rightarrow X \times I$ with π the normal bundle projection. It is not hard to see, using duality data in the category of free pointed simplicial $G(X)$ -sets, that the homotopy equivalence η is compatible with the involution τ_V . Since $N(M)$ is codimension 0 submanifold of \mathbb{R}^N , the bottom horizontal map in the diagram (19) induces the homomorphisms $\pi_* A^s(N(M)) \rightarrow \pi_* A(N(M))$ whose images are contained in $Inv_{\tau_V}^+ \pi_* A(N(M))$. These facts, together with the commutative diagram (19) implies the condition (2) is true for a general M and this completes the proof. \square

Proof of Theorem 1.1. Since the calculation of [Wal78, P.48] and [FH78] gives

$$Inv_{\tau_V}^+ \pi_{i+2}^{\mathbb{Q}} A(*) = Inv_{\tau_{BF}}^+ K_{i+2}(\mathbb{Z}) \otimes \mathbb{Q} = 0$$

for all $i \geq 0$ and

$$\dim Inv_{\tau_V}^- \pi_{i+2}^{\mathbb{Q}} A(*) = \dim Inv_{\tau_{BF}}^- K_{i+2}(\mathbb{Z}) \otimes \mathbb{Q} = \begin{cases} 1, & \text{if } i \equiv 3 \pmod{4}; \\ 0, & \text{otherwise,} \end{cases}$$

then Theorem 1.1 follows immediately from Theorem 2.1 and Lemma 3.1. \square

4 Application to the space of the nonnegatively curved metrics

Belegarde, Farrell and Kapovitch proved in [BFK15] that for many open manifolds V admitting complete nonnegatively curved Riemannian metrics and for certain positive integers i the rational homotopy groups $\pi_i^{\mathbb{Q}} \mathcal{R}_{K \geq 0}(V) \neq 0$, where $\mathcal{R}_{K \geq 0}(V)$ is the space of complete nonnegatively curved Riemannian metrics on V . For example, when $d \geq 2$ and for some explicit $i \geq 2$, they prove that there exists an m such that

$$\pi_i^{\mathbb{Q}} \mathcal{R}_{K \geq 0}(TS^{2d} \times S^m) \neq 0.$$

However, the methods in [BFK15] are insufficient to determine m exactly when i is given. It turns out, though, that this can be fixed by computing the ranks of $Inv^+ \pi_k^{\mathbb{Q}} \mathcal{P}(M)$ and $Inv^- \pi_k^{\mathbb{Q}} \mathcal{P}(M)$ when M is the unit sphere bundle of S^{2d} . In this paper, we will concentrate on the cases when $V = TS^{2d} \times S^m$ and the method to deal with the remaining cases is similar.

For a graded \mathbb{Q} -vector space $A_* = \bigoplus_i A_i$, let

$$\mathbf{p}(A_*) = \sum_i (\dim_{\mathbb{Q}} A_i) \cdot t^i$$

be the Poincaré Series of A_* . In this section, we will compute the Poincaré Series for $Inv^{\pm} H_{S^1}^*(LM; \mathbb{Q})$ and $Inv^{\pm} \pi_*^{\mathbb{Q}} \mathcal{P}(M)$. We begin with the following lemma.

Lemma 4.1. *Let X be a finite complex and let $H_{S^1}^*(LX; \mathbb{Q}) := H^*(ES^1 \times_{S^1} LX; \mathbb{Q})$. Then*

- (1) $\mathbf{p}(Inv^{\pm} H_{S^1}^*(LX, *, \mathbb{Q})) = \mathbf{p}(Inv^{\pm} H_{S^1}^*(LX; \mathbb{Q})) - \mathbf{p}(Inv^{\pm} H_{S^1}^*(*, \mathbb{Q}))$
- (2) $Inv^{\pm} H_{S^1}^*(LX; \mathbb{Q}) = Inv^{\pm} H_{S^1}^*(LX; \mathbb{Q})$
- (3) $\mathbf{p}(Inv^+ H_{S^1}^*(*, \mathbb{Q})) = \frac{1}{1-t^4}$ and $\mathbf{p}(Inv^- H_{S^1}^*(*, \mathbb{Q})) = \frac{t^2}{1-t^4}$

Proof. The constant map $M \rightarrow *$ induces a retraction $ES^1 \times_{S^1} LM \rightarrow ES^1 \times_{S^1} *$ which is compatible with the involution T , hence induces the decomposition

$$H_{S^1}^*(LM; \mathbb{Q}) \cong H_{S^1}^*(LM, *, \mathbb{Q}) \oplus H_{S^1}^*(*, \mathbb{Q})$$

which is compatible with the involution T_* as well and so this implies (1).

Formula (2) follows from the Universal Coefficient theorem.

Since $H_{S^1}^*(*, \mathbb{Q}) = H^*(BS^1; \mathbb{Q}) = \mathbb{Q}[\alpha]$ with $\dim \alpha = 2$ and the involution T_* is given by $\alpha \mapsto -\alpha$ (c.f. [KS88, THEOREM 3.3]), then (2) implies (3). \square

Let $\Lambda(x_1, \dots, x_l)$ denote the free graded algebra over \mathbb{Q} generated by x_1, \dots, x_l ; this algebra is the tensor product of the polynomial algebra generated by the even dimensional generators and the exterior algebra generated by the odd dimensional generators.

Example 4.2. Let $M = S(TS^{2d})$: the unit tangent bundle of S^{2d} for $d \geq 2$. The rational cohomology ring for M is the exterior algebra $\Lambda(a)$ on one generator a with $\dim a = 4d - 1$. Then the minimal model for M is $(\Lambda(x), \delta)$ where differential $\delta = 0$ and $\dim x = 4d - 1$. By [VPB85], the minimal model for the space $ES^1 \times_{S^1} LM$ is $(\Lambda(\alpha, x, \bar{x}), D)$, $\dim \alpha = 2$, $\dim x = 4d - 1$, $\dim \bar{x} = 4d - 2$ with differential $D\alpha = 0$, $D\bar{x} = 0$ and $Dx = \alpha\bar{x}$. It is not hard to check that $H_{S^1}^*(LM; \mathbb{Q})$ is generated freely as vector space by α^k, \bar{x}^l for all nonnegative integers k, l . As the involutions on $H_{S^1}^*(LM; \mathbb{Q})$ is given by $\alpha \mapsto -\alpha$, $\bar{x} \mapsto -\bar{x}$ (c.f. [KS88, THEOREM 3.3]), then

$$\begin{aligned} p(H_{S^1}^*(LM)) &= \frac{1}{1-t^2} + \frac{t^{4d-2}}{1-t^{4d-2}} \\ p(\text{Inv}^+ H_{S^1}^*(LM)) &= \frac{1}{1-t^4} + \frac{t^{8d-4}}{1-t^{8d-4}} \\ p(\text{Inv}^- H_{S^1}^*(LM)) &= \frac{t^2}{1-t^4} + \frac{t^{4d-2}}{1-t^{8d-4}} \end{aligned}$$

By the Theorem 1.1 and Lemma 4.1, one obtains

$$\begin{aligned} p(\text{Inv}^+ \pi_*^{\mathbb{Q}} \mathcal{P}(M)) &= \frac{t^3}{1-t^4} + \frac{t^{8d-5}}{1-t^{8d-4}} \\ p(\text{Inv}^- \pi_*^{\mathbb{Q}} \mathcal{P}(M)) &= \frac{t^{12d-7}}{1-t^{8d-4}} \end{aligned}$$

For a compact smooth manifold N let $P'(N)$ be the topological group of the diffeomorphisms of $N \times [0, 1]$ that are identity on a neighborhood of $N \times \{0\} \cup \partial N \times [0, 1]$. Assume further that $\partial N \neq \emptyset$ and identify $\partial N \times [0, 1]$ with a fixed collar neighborhood of N . Define $\iota_N : P'(\partial N) \rightarrow \text{Diff}(N)$ to be the map that extends every $f \in P'(\partial N)$ from the collar $\partial N \times [0, 1]$ to a diffeomorphism of N by taking the identity outside the collar. It follows from [BFK15, Theorems 1.1 and 1.2] that $\pi_{i+1}^{\mathbb{Q}} \mathcal{R}_{K \geq 0}(TS^{2d} \times S^m) \neq 0$ if $\ker \pi_i^{\mathbb{Q}}(\iota_{E \times S^m}) \neq 0$ where $d \geq 2$ and E is the associated disk bundle of TS^{2d} . We will find a condition for $\ker \pi_i^{\mathbb{Q}}(\iota_{E \times S^m}) \neq 0$ in terms of the positive and negative eigenspaces of the involution on $\pi_* \mathcal{P}(\partial E)$. For this, we restate [BFK15, Lemma 9.4] as follows.

Lemma 4.3. Let E be a compact manifold, m, i be integers such that $m \geq 0$, $i \geq 1$, $\dim \partial E + m \geq \max\{3i + 7, 2i + 9\}$ and

$$\frac{\dim \pi_i^{\mathbb{Q}} \mathcal{P}(\partial E)}{2} \leq \begin{cases} \dim \text{Inv}^+ \pi_i^{\mathbb{Q}} \mathcal{P}(\partial E), & \text{if } \dim \partial E + m \equiv 0 \pmod{2}; \\ \dim \text{Inv}^- \pi_i^{\mathbb{Q}} \mathcal{P}(\partial E), & \text{otherwise.} \end{cases}$$

Then

$$\dim \ker \pi_i^{\mathbb{Q}}(\iota_{E \times S^m}) \geq \frac{\dim \pi_i^{\mathbb{Q}} \mathcal{P}(\partial E)}{2} - \dim \pi_i^{\mathbb{Q}} \text{Diff}(E \times D^m, \partial)$$

Proof. Let $\eta_i : \pi_i^{\mathbb{Q}} P(\partial E \times D^m) \rightarrow \pi_i^{\mathbb{Q}} P(\partial E \times D^m)$ be given by

$$\eta_i(x) = x + \iota_*(x)$$

with ι the involution on $P(\partial E \times D^m)$. Since $\ker \eta_i = \text{Inv}^- \pi_i^{\mathbb{Q}} P(\partial E \times D^m)$ and

$$\dim \pi_i^{\mathbb{Q}} P(\partial E \times D^m) = \dim \text{Im} \eta_i + \dim \ker \eta_i,$$

then by assumption one gets

$$\dim \text{Im} \eta_i = \dim \text{Inv}^+ \pi_i^{\mathbb{Q}} P(\partial E \times D^m) \geq \frac{\dim \pi_i^{\mathbb{Q}} \mathcal{P}(\partial E)}{2} \quad (20)$$

Since the inclusion $P'(M) \rightarrow P(M)$ is a weak homotopy equivalence (c.f. [Igu88, Chapter 1, Proposition 1.3]), then the arguments in [BFK15, p.11] imply that the image of the $\pi_i^{\mathbb{Q}}$ -homomorphism induced by the inclusion $j : P'_\partial(\partial E \times D^m) \rightarrow P'(\partial E \times D^m)$ has dimension not less than $\dim \text{Im} \eta_i$, hence by the inequality (20) we have

$$\dim \text{Im} \pi_i^{\mathbb{Q}}(j) \geq \frac{\dim \pi_i^{\mathbb{Q}} \mathcal{P}(\partial E)}{2}.$$

Then the same argument as in the proof of [BFK15, Lemma 9.4] completes the proof. \square

Example 4.4. Let E be the associated disk bundle of TS^{2d} . By [BFK15, Corollary 8.5, Proposition 9.1 and Theorem 9.11], a sufficient condition for $\dim \pi_i^{\mathbb{Q}} \text{Diff}(E \times D^m, \partial) = 0$ and $\dim \pi_i^{\mathbb{Q}} \mathcal{P}(\partial E) = 1$ is as follows:

$$\begin{cases} i = 8d - 5 + (4d - 2)j \text{ for some odd } j \geq 1 \\ 3i + 9 < 4d + m \\ m + i \geq 4d \\ m \equiv 3 \text{ or } 2d \pmod{4} \end{cases}$$

This, together with the condition Lemma 4.3 gives a sufficient condition for $\ker \pi_i^{\mathbb{Q}}(\iota_{E \times S^m}) \neq 0$, which is

$$\begin{cases} i = 8d - 5 + (4d - 2)j \text{ for some odd } j \geq 1 \\ 3i + 9 < 4d + m \\ m + i \geq 4d \\ m \equiv 3 \text{ or } 2d \pmod{4} \\ \frac{\dim \pi_i^{\mathbb{Q}} \mathcal{P}(\partial E)}{2} \leq \begin{cases} \dim \text{Inv}^+ \pi_i^{\mathbb{Q}} \mathcal{P}(\partial E), & \text{if } \dim \partial E + m \equiv 0 \pmod{2}; \\ \dim \text{Inv}^- \pi_i^{\mathbb{Q}} \mathcal{P}(\partial E), & \text{otherwise.} \end{cases} \end{cases} \quad (21)$$

By the calculation of Example 4.2 we get

$$p(\text{Inv}^+ \pi_*^{\mathbb{Q}} \mathcal{P}(\partial E)) - p(\text{Inv}^- \pi_*^{\mathbb{Q}} \mathcal{P}(\partial E)) = \frac{t^3}{1 - t^4} + \sum_m (-1)^m t^{(4d-2)m+8d-5}$$

Since $(4d - 2)m + 8d - 5 \equiv 1 \pmod{4}$ when m is odd, then a necessary and sufficient condition for $\dim \text{Inv}^- \pi_i^{\mathbb{Q}} \mathcal{P}(\partial E) > \dim \text{Inv}^+ \pi_i^{\mathbb{Q}} \mathcal{P}(\partial E)$ is $i = (4d - 2)m + 8d - 5$ for some odd m . Consequently, the condition (21) can be simplified as

$$\begin{cases} i = 8d - 5 + (4d - 2)j \text{ for some odd } j \geq 1 \\ m > 20d - 6 + (12d - 6)j \\ m \equiv 2d \pmod{4} \end{cases} \quad (22)$$

For example, when $d = 2$, the first i and m appearing here are $(i, m) = (17, 56), (29, 92), (41, 128)$, which give $\pi_{18}^{\mathbb{Q}} \mathcal{R}_{K \geq 0}(TS^4 \times S^{56}) \neq 0$, $\pi_{30}^{\mathbb{Q}} \mathcal{R}_{K \geq 0}(TS^4 \times S^{92}) \neq 0$, $\pi_{42}^{\mathbb{Q}} \mathcal{R}_{K \geq 0}(TS^4 \times S^{128}) \neq 0$.

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